Answers To a Selection of Problems from

Classical Electrodynamics

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2.23 A Hollow Cubical Conductor

a. The Potential inside the Cube

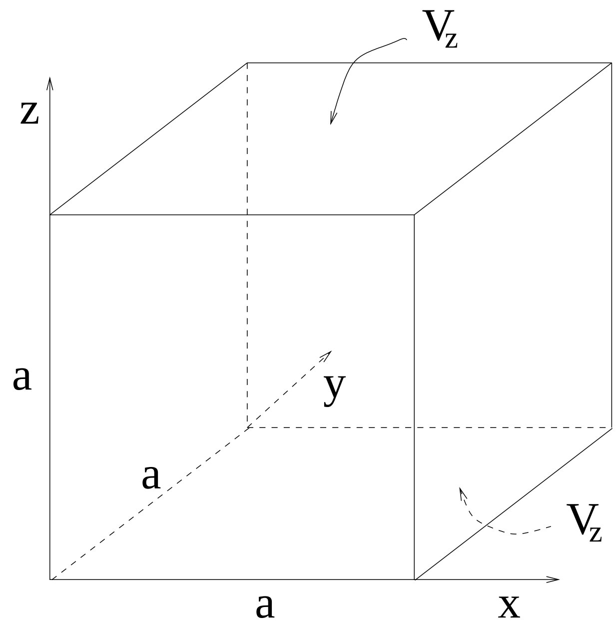

Figure 2.9: A hollow cube, with all sides but z=0 and z=a grounded.

$$\nabla^2\Phi = 0 \tag{2.53}$$

Separating the variables:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0. \tag{2.54}$$

x and y can vary independently so each term must be equal to a constant $-\alpha^2$:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \alpha^2 = 0 \quad \Rightarrow \quad X = A\cos\alpha x + B\sin\alpha x \tag{2.55}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} + \beta^2 = 0 \quad \Rightarrow \quad Y = C\cos\beta y + D\sin\beta y \tag{2.56}$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} + \gamma^2 = 0 \quad \Rightarrow \quad Z = E\sinh(\gamma z) + F\cosh(\gamma z), \tag{2.57}$$

where $\gamma^2 = \alpha^2 + \beta^2$. The boundary conditions determine the constants:

$$\Phi(0, y, z) = 0 \quad \Rightarrow \quad A = 0$$

$$\begin{aligned}
\Phi(a,y,z)=0 &\quad\Rightarrow\quad \alpha_n = n\pi/a \quad (n=1,2,3,...)\\
\Phi(x,0,z)=0 &\quad\Rightarrow\quad C=0\\
\Phi(x,a,z)=0 &\quad\Rightarrow\quad \beta_m = m\pi/a \quad (m=1,2,3,...)\\
&\quad\Rightarrow\quad \gamma_{nm} = \pi\sqrt{n^2+m^2}
\end{aligned}$$

The solution is thus reduced to

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[A_{nm}\sinh\left(\frac{\gamma_{nm}z}{a}\right) + B_{nm}\cosh\left(\frac{\gamma_{nm}z}{a}\right)\right] \tag{2.58}$$

Now, let's use the last boundary conditions to find the coefficients A_nm and B_nm. The top and bottom of the cube are held at a constant potential V_z, so

$$\Phi(x,y,0) = V_z = \sum_{n,m=1}^{\infty} B_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right) \tag{2.59}$$

This means that B_{nm} are merely the coefficients of a double Fourier series (see for instance [1] on Fourier series):

$$B_{nm} = \frac{4V_z}{a^2}\int_0^a\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dxdy \tag{2.60}$$

It can be easily shown that the individual integrals in equation (2.60) are zero for even integer values and $\frac{2a}{n\pi}$ for n is odd. Thus B_nm is

$$B_{nm} = \frac{16V_z}{\pi^2 nm} \qquad \text{for odd } (n,m) \tag{2.61}$$

The top of the cube is also at constant potential V_z, so

$$\begin{aligned}
\Phi(x,y,0) &= V_z = \Phi(x,y,a) \Leftrightarrow\\
B_{nm} &= A_{nm}\sinh(\gamma_{nm}) + B_{nm}\cosh(\gamma_{nm}) \Leftrightarrow\\
A_{nm} &= B_{nm}\frac{1-\cosh(\gamma_{nm})}{\sinh(\gamma_{nm})}
\end{aligned} \tag{2.62}$$

Substituting the expressions for A_{nm} and B_{nm} into equation (2.58), gives us

$$\begin{aligned}
\Phi(x,y,z) = \frac{16V_z}{\pi^2}\sum_{n,m\ odd}^{\infty}\frac{1}{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)&\left[\frac{1-\cosh(\gamma_{nm})}{\sinh(\gamma_{nm})}\sinh\left(\frac{\gamma_{nm}z}{a}\right)\right.\\
&\left.+\cosh\left(\frac{\gamma_{nm}z}{a}\right)\right],
\end{aligned} \tag{2.63}$$

where $\gamma_{nm} = \pi\sqrt{n^2+m^2}$.

b. The Potential at the Center of the Cube

The potential at the center of the cube is

$$\Phi(\frac{a}{2},\frac{a}{2},\frac{a}{2}) = \frac{16V_z}{\pi^2}\sum_{n,m\ odd}^{\infty}\frac{1}{nm}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right)\left[\frac{1-\cosh(\gamma_{nm})}{\sinh(\gamma_{nm})}\sinh\left(\frac{\gamma_{nm}}{2}\right) + \cosh\left(\frac{\gamma_{nm}}{2}\right)\right] \tag{2.64}$$

With just $n, m = 1$, the potential at the center is

$$\frac{16V_z}{\pi^2}\left[\frac{1-\cosh(\sqrt{2}\pi)}{\sinh(\sqrt{2}\pi)}\sinh\left(\frac{\pi}{\sqrt{2}}\right) + \cosh\left(\frac{\pi}{\sqrt{2}}\right)\right] \approx 0.347546V_z \tag{2.65}$$

When we add the two terms $(n = 3,\ m = 1)$ and $(n = 1,\ m = 3)$, the potential is $0.332498V_z$.

c. The Surface Charge Density

The surface charge density on the top surface of the cube is given by

$$\sigma = -\epsilon_0\frac{\partial\Phi}{\partial z}\bigg|_{z=a} \tag{2.66}$$

In the appendix it is shown that the differentiation of the hyperbolic sine is the hyperbolic cosine. Furthermore

$$\frac{d\cosh(az)}{dz} = a\sinh(az) \tag{2.67}$$

Using this equality in differentiating the expression for the potential in equation (2.63), we get

$$\frac{\partial\Phi}{\partial z} = \frac{16V_z}{\pi^2}\sum_{n,m\ odd}^{\infty}\frac{\gamma_{nm}}{nma}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[\frac{1-\cosh(\gamma_{nm})}{\sinh(\gamma_{nm})}\cosh\left(\frac{\gamma_{nm}z}{a}\right) + \sinh\left(\frac{\gamma_{nm}z}{a}\right)\right], \tag{2.68}$$

where $\gamma_{nm} = \pi\sqrt{n^2+m^2}$. Now we evaluate this expression for $z = a$:

$$\sigma = -\epsilon_0\frac{\partial\Phi}{\partial z}\bigg|_{z=a} =$$

$$-\frac{16\epsilon_0V_z}{\pi^2}\sum_{n,m\ odd}^{\infty}\frac{\gamma_{nm}}{nma}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[\frac{1-\cosh(\gamma_{nm})}{\sinh(\gamma_{nm})}\cosh(\gamma_{nm}) + \sinh(\gamma_{nm})\right] =$$

$$-\frac{16\epsilon_0V_z}{\pi^2}\sum_{n,m\ odd}^{\infty}\frac{\gamma_{nm}}{nma}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[(1-\cosh(\gamma_{nm}))\coth(\gamma_{nm}) + \sinh(\gamma_{nm})\right] \tag{2.69}$$

Further simplification??